REAL AND DISCRETE HOLOMORPHY: INTRODUCTION TO AN ALGEBRAIC APPROACH

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Abstract. We consider spaces for which there is a notion of harmonicity for complex valued functions defined on them. For instance, this is the case of Riemannian manifolds on one hand, and (metric) graphs on the other hand. We observe that it is then possible to define an "amusing" notion of holomorphic functions on them, and show how rigid it is in some cases.

Resumé. On peut parler d'harmonicité des fonctions à valeurs complexes, définies sur des variétés riemanniennes, ou sur des graphes métriques, ainsi que sur d'autres espaces plus généraux. Nous observons ici, que dans tous ces cas, on peut aussi parler d'holomorphie de ces fonctions. Nous décortiquons le cas classique des variétés riemanniennes, et la cas discret de certains graphes. Nous montrons, en particulier, une rigidité dans le cas discret, qui le distingue du cas classique.

1. Introduction

1.1. **Harmonic and holomorphic.** It is a usual practice to compare harmonicity with holomorphy in any context where they are simultaneously defined. In the classical case of real and complex valued mappings defined on open subsets of $\mathbb{C}$, the dictionary is realized, locally, by means of the notion of conjugate part.

In the general case of hermitian complex manifolds, since one notion is metric, and the other is differentiable, the comparison is available, only if some compatibility between the metric and the complex structure is fulfilled. Roughly speaking this corresponds to the Kaehler property.

1.1.1. *General framework, Support of harmonicity.* Classically, harmonic functions are defined by means of a metric on a Riemannian manifold. However, there is larger class of spaces where one can talk about them. A substantial class is that of (measurable) spaces endowed with Markov processes. As particular interesting cases, we have random walks on groups, and graphs endowed with their natural "simplicial" Markov chains. Recall here that for a function f, its Laplacian Δf is the function:

$$\Delta f(x) = \frac{1}{\nu(x)} \sum_{y \sim x} f(y) - f(x)$$

where $y \sim x$ means y is adjacent to x, and $\nu(x)$ is the valency of x.

• In this paper, we will always deal with complex valued functions $\phi : X \to \mathbb{C}$, where X is *a Riemannian manifold* or *a graph*.

1.1.2. *Holomorphic functions.* Even in the most classical case, there are some and in fact deep differences between harmonicity and holomorphy. Maybe, the silliest one is that the (usual) product of two holomorphic functions is holomorphic, but this is not the case of harmonic functions!

This suggests to define holomorphic functions just by forcing invariance under multiplication:

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Definition 1.1. *Let $\phi : X \to \mathbb{C}$, where X is a Riemannian manifold or a graph. We say that ϕ is holomorphic if both ϕ and its square ϕ^2 are harmonic.*

Recall the classical formula for functions on Riemannian manifolds:

$$\Delta(fg) = f\Delta g + g\Delta f + \langle \nabla f, \nabla g\rangle$$

(it is remarkable that this is also valid in the case of graphs, for a natural definition of the gradient ∇).

Therefore, if $X = \mathbb{R}^2$ ($= \mathbb{C}$), $\phi = f + \sqrt{-1}g$ is holomorphic in the sense of the definition above, iff, $\| \nabla f \| = \| \nabla g \|$, and $\langle \nabla f, \nabla g\rangle = 0$. This exactly means that ϕ is conformal, or to use a more precise terminology, it is *semi-conformal* which means that ϕ may have singular points (i.e. where $d\phi = 0$).

1.1.3. *Some remarks.*

• A mapping $\phi : \mathbb{C} \to \mathbb{C}$ is holomorphic, according with our definition, iff ϕ is holomorphic or anti-holomorphic, in the usual sense.

Actually, an individual anti-holomorphic mapping (on any complex manifold) can be restored to become holomorphic. However, when considering all holomorphic and anti-holomorphic mappings together, it is a nuisance to separate them. Indeed that is exactly the counterpart of non-linearity in the equation $\Delta\phi = \Delta\phi^2 = 0$. In the general case unlike that of complex manifolds, this does not split into two linear equations corresponding to holomorphy and anti-holomorphy.

However, for the sake of simplicity, we will keep our term "holomorphic" as it is defined, since in fact, we will always deal with individual functions.

• In the classical case of a mapping $\phi : \mathbb{C} \to \mathbb{C}$, conformality of ϕ, implies conformality and hence harmonicity for all powers ϕ^N, for all integer N, that is, for all N, the equation

$$\Delta\phi = \ldots = \Delta\phi^N = 0 \qquad (*)_N$$

is satisfied (this has an obvious meaning for $N = \infty$). More generally:

Theorem 1.2.

• *For any Riemannian manifold X, we have,*

$$(*)_2 \Longrightarrow (*)_\infty$$

That is, a holomorphic function satisfies all $(*)_N$ for any N.

• *For any $N > 2$, there is a graph X and a holomorphic function on it, which satisfies $(*)_N$ but which does not satisfy $(*)_{N+1}$.*

• *For a graph X, of finite valency, there is a finite $m = m(X)$, such that any function satisfying $(*)_m$ is constant (and hence satisfies trivially $(*)_\infty$).*

1.1.4. *A f urther motivation : Dirichlet problem.* Consider the simplest Dirichlet problem for *bounded* harmonic functions defined on the unit disc $\mathbb{D}$ of $\mathbb{C}$. Classical harmonic analysis theory allows one to define a boundary value isomorphism :

$$\phi \in h^\infty(\mathbb{D}) \to \partial_\infty\phi \in L^\infty(\partial\mathbb{D})$$

from the Hardy space of bounded harmonic functions on the disc, to the space of bounded measurable functions on the circle.

The Hardy space $H^\infty(\mathbb{D})$ of holomorphic (in the usual sense) bounded functions, is a part of $h^\infty(\mathbb{D})$, but its image in $L^\infty(S^1)$ by the boundary value mapping, is very hard to explicit. In analytic words, this certainly reduces to an invariance by Hilbert transform, but this cannot help in understanding how holomorphic functions are more "regular" than harmonic ones. It is singularly suggesting to bring out a criterion, ensuring that a function $\psi \in L^\infty(S^1)$ is the boundary value of a holomorphic function!

Therefore, our motivation from the Dirichlet problem viewpoint, is that we are introducing here an "abstract" notion of holomorphy, which may help to understand the true nature of holomorphic functions. On the other hand, this notion of holomorphy for functions on X, allows one to specify a class of bounded measurable functions on its Poisson boundary, which are more regular than the others.

1.1.5. *A more general framework.* Actually, one can take any ring F, and consider F-valued functions defined on X. In the case of graphs, it is meaningful to speak of harmonicity. As for Riemannian manifolds, one just needs the ring to be topological, since the Laplacian can be defined via its infinitesimal mean value property.

Definition 1.3. *Let X be a Riemannian manifold or a graph, and F a topological ring. A mapping $\phi : X \to F$ is called F-holomorphic, iff ϕ and ϕ^2 are harmonic.*

We have just rewritten the same definition given in the case $F = \mathbb{C}$. Our goal is to point out some examples that seem exciting, and deserve attention. We mention here the case when F is, the quaternionic field (over the reals), and the matrix algebras $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$...

Remark 1.4. As an example, in another direction, consider the D'Alembertian operator on $\mathbb{R}^2$, $\Box\phi = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\phi$, where $\phi : \mathbb{R}^2 \to \mathbb{R}^2$. See the second $\mathbb{R}^2$ as a ring (in fact an $\mathbb{R}$-algebra) which is the product of $\mathbb{R}$ by itself. Suppose ϕ is a solution of $\Box\phi = \Box\phi^2 = 0$. It then follows, essentially (up to a switch of factors...), that $\phi(x, y) = (f(x), g(y))$. This means that ϕ is a conformal mapping of the Minkowski space $(\mathbb{R}^2, dxdy)$.

Remark 1.5. The previous example suggests to define a wave equation on graphs. Recall at this stage, that more generally than the canonical Markov chain on a graph, there are Markov chains with arbitrary non-equiprobable transition functions, or equivalently, the graph has weighted edges. This is also equivalent to endow the graph with an adapted metric, which is just given by these weights.

Now, the idea is to allow weights to take negative values. Analogously to the term "pseudo-Riemannian", one may call such graphs "pseudo-metric" As a diffusion operator, one obtains something like a D'Alembertian, with more complicated properties. They seem well adapted and connected to the dynamics on the graph. It is particularly interesting to define, amongst pseudo-metric graphs, a notion of Lorentzian ones: there is only one negative direction which corresponds to the "time".

1.2. **Harmonic morphisms.** The discussion above corresponds essentially to our chronological motivation in bringing out this notion of holomorphy. We then looked for possible correlated notions in the geometric literature. It was difficult to conclude if we weren't, hopefully, intuitively guided to harmonic morphisms. After, having a look on this wide theme, we discovered that our holomorphic functions on Riemannian manifolds, are nothing but harmonic morphisms with values in $\mathbb{R}^2$. This is however obscure, and not clear to extract, even, in standard celebrated references in this theory. In any case, motivation of harmonic morphisms are never stated as ours here. On the other hand, in the discrete case, that is for graphs, a theory of harmonic morphisms is not yet completely established. Maybe, the problem comes from the choice, to see Riemannian manifolds and graphs as objects of the same category or not. Anyway, let's anticipate and say that our holomorphic functions can not be seen as harmonic morphisms from graphs to $\mathbb{R}^2$.

1.2.1. *Riemannian case.* Recall that for Riemannian manifolds, a harmonic morphism is a mapping which preserves sheaves of local harmonic real functions. In other words, $\phi : X \to Y$ is a harmonic morphism, if for any real harmonic function f defined on an open subset of Y, $f \circ \phi$ is harmonic.

1.2.2. *Graph case.* Obviously in this definition, one can let X and Y, to be, both, or just one among then, a graph (instead of a Riemannian manifold). In particular, one can speak of harmonic morphisms $X \to \mathbb{C}$, where X is a graph.

1.2.3. *A difference.* Harmonic morphisms in the discrete case do not behave as nicely as in the classical case. Here is one difference between the two situations, with respect to our notions of holomorphy.

Theorem 1.6. *Let $\phi : X \to \mathbb{C}$ be a mapping.*

- *If X is a Riemannian manifold, then ϕ is holomorphic, iff, ϕ is a harmonic morphism.*
- *There is a graph X having a holomorphic function $\phi : X \to \mathbb{C}$ which is not a harmonic morphism.*

In fact, a non-constant harmonic morphism is an open mapping (see §2). In particular a harmonic morphism from a 1-dimensional objet (for instance a graph) to $\mathbb{C}$ is constant. The second part of the theorem above means that there are non-constant holomorphic functions on some graphs.

1.3. **Content of the article, Rigidity in the discrete case.** Our first aim is to introduce this point of view of holomorphy, which we find sufficiently motivated as we said above, and it is at least amusing! This point of view is obscure in the literature related to harmonic morphisms. For example, (a) Jacobi's problem, a precursor problem in the theory, since it exactly asks for a classification of global harmonic morphisms $\mathbb{R}^3 \to \mathbb{R}^2$, is expressed as a partial differential equation, but different from our formulation of holomorphic mappings (from $\mathbb{R}^3 \to \mathbb{C}$). This problem was solved only recently [1]. Other more "elementary" proofs were obtained afterwards (see for instance [3]). We anticipate here to announce our "elementary" proof which will appear in a forthcoming paper.

Also, as a contribution of the paper is the fact that the concept of holomorphy in the discrete case seems to be new. We were convinced of this for a long time, and after parts of the present paper were written, we discovered (in special circumstances) that the terminology "discrete holomorphy" was already used. Indeed in [6] (and related papers) the author introduced a notion of holomorphy for forms and functions on graphs. Inspired by Hodge theory, the definition is global, and linear. It is therefore different from our notion, yet, possible analogies are interesting to find.

1.3.1. *Content.* Instead of a systematic study of discrete holomorphy in the present article, we focus on some particular examples which illustrate various subtleties.

We will give some attention to the simplest non-trivial cases : the 3-valenced tree T_3 and its (1-dimensional) "dual graph" Tr_3 (see §6). The spaces of holomorphic functions have finite dimension (which is of course not the case of harmonic functions), and endowed with the beautiful action of the automorphism group.

1.3.2. *Random dynamics versus Poisson kernel.* In fact, holomorphic functions are "essentially" determined when given on some finite subsets of vertices. There is a "holomorphic dynamics" which allows one to calculate the holomorphic functions progressively on the other vertices. However, there is at each step some finite choice to consider, in some sense, one has a kind of "random dynamics".

One can conclude some general philosophy. In the case of harmonic functions, one goes to (space) infinity, and reproduce objects by a Poisson integral. In the holomorphic case, a random holomorphic dynamics of finite subsets of the space itself allows one to reproduce the function.

We do not claim our tentative to formulate all these notions is optimal. We find however exciting (and funny) the beauty of these structures and their connections with many natural notions, as correspondences ([2]), ...

Let us now summarize some obtained results.

1.3.3. *The 3-valenced tree T_3.* The case of T_3 is the simplest non-trivial one. Many (amusing) facts will be said about it. The following theorem summarizes results obtained along §4.

Theorem 1.7. *Let, $\phi \in Hol^*(T_3)$, that is, a non-constant holomorphic function on T_3. Then:*

(1) *ϕ sends (the vertices of) T_3 onto (the vertices of) a hexagonal tiling τ of the Euclidean plane.*
(2) *In fact ϕ is the universal covering of the hexagonal tiling. In particular:*
   • *ϕ is locally injective, that is, it is injective on any (closed) ball of radius 1 in T_3. More precisely, ϕ is conformal (see Remark 4.1 for definition).*
   • *the image of a geodesic of T_3 is a locally injective walk on the hexagonal tiling. It could be periodic, but generally goes to infinity.*
   — *Let $\mathcal{H}_{\alpha,\beta}(T_3)$, for $\alpha \neq \beta$ be the space of holomorphic functions taking the values α and β on two adjacent vertices. Let H be the subgroup of $Aut(T_3)$ fixing (individually) these vertices, then:*
(3) *The action of H on $\mathcal{H}_{\alpha,\beta}(T_3)$ is simply transitive. In particular, the group and the space are homeomorphic (in particular both are compact).*
(4) *The same is true for the product action of the group $SG \times H$ on the space $Hol^*(T_3)$, where SG is the similarity group of $\mathbb{C}$. The previous product group and space are therefore homeomorphic.*
(5) *In contrast, the (transitive) action of $SG \times Aut(T_3)$ on $Hol^*(T_3)$ is not free. The stabilizer of a point (i.e. a non-constant holomorphic function) is a discrete group Γ acting freely on T_3, which is nothing but the fundamental group of the hexagonal tiling.*

One sees from this how are rigid holomophic functions in comparison to harmonic ones. In particular, the space of holomorphic functions has finite (topological) dimension (exactly 4), whereas that of harmonic has infinite dimension.

Also, one sees relationship between holomorphy and random walk on the hexagonal lattice, in fact, only partial random walks are involved, those, called here locally injective.

1.3.4. *The graph Tr_3.* Here, a holomorphic function is determined once given on any triangle, but with some discrete indetermination, leading to a holomophic dynamics on the space of triangles.

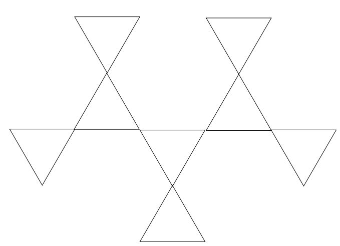

Figure 1 : The graph Tr_3

Theorem 1.8. *A holomorphic function on Tr_3 is determined once given on any one of its triangles. There is however some discrete indetermination in the extending process of the holomorphic function, leading to a holomophic random dynamics in the space of triangles.*

• Correspondence. *Precisely, this holomorphic dynamics is apprehended in the correspondence $\mathcal{S}$ which is a 3-dimensional (complex) quadric defined in $\mathbb{C}^3 \times \mathbb{C}^3$ (endowed with coordinates $(p, e, f; x, y, z)$) by:*

$$\begin{aligned} e + f - y + (-y + z) &= 0 \\ e^2 + f^2 + y^2 + (-y + z)^2 &= 0 \\ p - (x + y) &= 0 \end{aligned}$$

• Random dynamics. *In other words, $\mathcal{S}$ is the graph of a holomorphic multi-valued transformation $M : (p, e, f) \in \mathbb{C}^3 \to (x, y, z) \in \mathbb{C}^3$, which encodes the diffusion process of a holomorphic function.*

• Action of $Aut(Tr_3)$. *Two holomorphic functions are equal up to composition by an element of $Aut(Tr_3)$, iff, they take same values on vertices of two triangles of Tr_3.*

• Orbital structure. *On the space of marked triangles $\mathbb{C} \times (\mathbb{C}^2 - \{0\})$, consider the relation: $\mathbf{\Delta} \sim \mathbf{\Delta}' \iff \exists\, \phi$ holomorphic such that $\mathbf{\Delta}' = \phi(\mathbf{\Delta})$. Then $\sim$ is an equivalence relation.*

• Projectivization. *By homogeneity, we get a correspondence in $\mathbb{C}P^1 \times \mathbb{C}P^1$, and an equivalence relation in $\mathbb{C}P^1$.*

(It turns out that the equivalence relation $\sim$, or its projective variant, are not defined by group actions).

These two theorems do not answer many other remaining questions on the subject of holomorphic functions on these special two spaces T_3 and Tr_3. Actually, in this introductory article, we could not go deeply in analysis : the phenomenology by itself is rich enough, and formulations are not straightforward! Preliminary simulations promise a beautiful and strong dynamics.

1.3.5. *Other graphs, non-rigid situations.* The results for T_3 allow one to understand holomorphic functions for any 3-valenced graph. This is discussed in §7, where also remarks are made, giving evidences to a rigidity phenomenon in the case of the Cayley graph of $\mathbb{Z}^2$. In contrast, one observes abundance of holomorphic functions for trees of valency ≥ 4. The study of these trees will be continued in §8, where we formulate the "conjugate part problem".

2. Some properties of harmonic morphisms, Proofs in the Riemannian case

We recall in what follows, fundamental properties of harmonic morphisms in the context of Riemannian manifolds. They are due to B. Fuglede and T. Ishihara (independently) [4, 5]. Let $\phi : X \to Y$ be a smooth mapping between two Riemannian manifolds. Then *ϕ is a harmonic morphism, iff, ϕ is harmonic and horizontally semi-conformal.* The latter means that, if ϕ is not constant, then for almost all x, if H_x denotes the normal space of $\ker D_x\phi$, then: $D_x\phi : H_x \to T_{\phi(x)}Y$ is a conformal isomorphism (on the complementary negligible set where $D_x\phi = 0$).

It then follows in particular that a harmonic morphism is an open mapping.

As an example if $\dim X = 1$, and $\dim Y = 2$, then the harmonic morphism ϕ is constant.

Remark 2.1. This fact can also be proved (in an essentially similar way) when X is merely a graph (say, a 1-dimensional object more general than manifolds). This explains the comment after Theorem 1.6.

Consider now a mapping $\phi : X \to \mathbb{C}$, where X is a Riemannian manifold.

Assume ϕ is a harmonic morphism. The mappings $z \to z^N$ are harmonic on $\mathbb{C}$. By definition, ϕ^N is harmonic for any N, in particular ϕ is holomorphic.

Assume now that ϕ is holomorphic. From the discussion following Definition 1.1, it follows that ϕ is horizontally semi-conformal (as defined above). ¿From the above characterization, ϕ is a harmonic morphism.

Summarizing, ϕ is a harmonic morphism, iff, $\Delta\phi = \Delta\phi^2 = 0$, iff, $\Delta\phi^N = 0$ for all N.

This proves the Riemannian parts in Theorems 1.2 and 1.6

Finally, let us say that the fact that harmonic morphisms satisfy a quadratic equation, or equivalently that $(*)_2 \implies (*)_\infty$, is stated through standard papers like [4]. However, it was just used as an intermediate step of proofs.

3. Generalities on graphs

We start with a general local study, and fix notations. Let s_0 be a vertex of X, with valency n: $s_1, \dots, s_n$ are the adjacent vertices of s_0. The formel difference $s_i - s_0$ denotes the oriented edge joigning s_0 to s_i.

Let ϕ be a (complex) function on X, with $z_i = \phi(s_i)$. The oscillation of ϕ along the edge $s_i - s_0$ is $\delta_i = z_i - z_0$. The gradiant of ϕ at s_0 is the vector $\overrightarrow{\delta} = (z_1 - z_0, \dots, z_n - z_0) \in \mathbb{R}^n$. (Actually the values $\{z_i\}$ are not ordered and therefore, $\overrightarrow{\delta}$ is an element of the symmetric quotient of $\mathbb{R}^n$. However this precision does not matter in our analysis here, and for the sake of simplicity, we keep our notations above).

The function ϕ is harmonic at s_0 if its gradiant at s_0 has a vanishing divergence (i.e. arithmetic mean): $\sum(z_i - z_0) = \sum \delta_i = 0$.

Fact 3.1. *The function ϕ is holomorphic, iff, both its mean oscillation and quadratic mean oscillation vanish at any vertex:*

$$\sum \delta_i = \sum \delta_i^2 = 0$$

In fact, more generally, for a harmonic function ϕ, the mean quadratic oscillation equals the mean oscillation of its square ϕ^2.

Proof. At s_0, the mean oscillation of ϕ and ϕ^2 are respectively: $\sum \delta_i = \sum(z_i - z_0)$ and $\sum(z_i^2 - z_0^2)$.

Assume ϕ is harmonic at s_0, then, $\sum z_i = nz_0$ (n is the valency at s_0). Thus,

$$\sum \delta_i^2 = \sum(z_i - z_0)^2 = \sum z_i^2 + nz_0^2 - 2z_0 \sum z_i = \sum z_i^2 + nz_0^2 - 2nz_0^2 = \sum(z_i^2 - z_0^2)$$

□

We will also need the following lemma. Its proof reduces to the possibility of solving in $\mathbb{C}$, a system of a linear and a quadratic equations, on two unkowns.

Lemma 3.2. *Let $n \geq 3$ and $\delta_1, ..., \delta_{n-2} \in \mathbb{C}$ be given. Then there exist a pair $\{\delta_{n-1}, \delta_n\}$, unique up to switch, of (complex) solutions of*

$$(\sum_{i=1}^{n-2} \delta_i) + \delta_{n-1} + \delta_n = (\sum_{i=1}^{n-2} \delta_i^2) + \delta_{n-1}^2 + \delta_n^2 = 0.$$

In particular, to fix notations for future use in §§4 and 6:

Fact 3.3. *For given e (resp. e and f) there is $\{u, v\}$ a unique pair up to switch, such that:*

$$\begin{aligned} e + u + v &= 0 \\ e^2 + u^2 + v^2 &= 0 \end{aligned}$$

(resp.

$$\begin{array}{rcl} e+f+u+v & = & 0 \\ e^2+f^2+u^2+v^2 & = & 0\ ) \end{array}$$

4. Dynamical and ergodic study for the 3-valenced tree T_3

We investigate here holomorphic functions on T_3, the (bi-infinite) tree of valency 3. However, because of technical difficulties (due to the fact that T_3 can not be well ordered, that is endowed with an orientation inducing an orientation on any geodesic), we will start by considering intermediate cases.

4.1. **The smallest tree, the tripod.** This is a (finite) tree $\mathbf{Y}$ with 4 vertices, O, O', A, B, all of valency 1, except O, which has valency 3 (all the others are extremal).

Let ϕ be a function on $\mathbf{Y}$, and denote by e, u, v its associated oscillations at O, that is $e = \phi(O') - \phi(O)$, $u = \phi(A) - \phi(O)$ and $v = \phi(B) - \phi(O)$.

Suppose ϕ is holomorphic at O, then,

$$e+u+v = e^2+u^2+v^2 = 0.$$

Fact 3.3 says that u and v are completely determined, up to order, when e is given. More precisely, let $j = e^{\frac{2}{3}\pi i}$ be a cubic root of unity, then, necessarily, up to order,

$$u = je, \quad v = j^2 e$$

For instance, $\phi(A) - \phi(O) = \alpha(\phi(O) - \phi(O'))$ for some $\alpha \in \{-j, -j^2\}$.

Thus,

$$\phi(A) = \phi(O) + \alpha(\phi(O) - \phi(O'))$$

Also, $\phi(A) = \phi(O') + (\phi(O) - \phi(O')) + \alpha(\phi(O) - \phi(O')$.

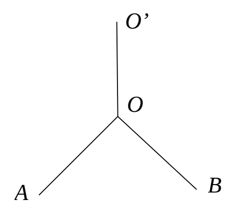

Figure 2 : The tripod

Remark 4.1. *(Conformality)* The natural "geometric" way to embed $\mathbf{Y}$ in the Euclidean plane is to imagine the edges OO', OA and OB, having the same length and making equal angles at O, which must be $\frac{2\pi}{3}$. We see from above, that a holomorphic mapping ϕ is then conformal (the target space $\mathbb{C}$ is of course endowed with its Euclidean structure).

4.2. **Rooted tree.** Let now $\mathbf{A}$ be an infinite tree of valency 3, with a root R. So, all vertices have valency 3, except R which has valency 2. A model of it is given by the free monoid on two letters $\{a, b\}$, the root R corresponds to the void word. The vertices are words on a and b.

Instead of $\mathbf{A}$, we consider an extended tree $\mathbf{A}_{O'O}$, obtained by gluing an oriented segment $O'O$ to $\mathbf{A}$, where O is identified with R. Now, the root becomes O'.

If ϕ is a function given at O' and O, then one extends it holomorphicaly step by step, applying the rule described for $\mathbf{Y}$. To formulate this, orient $\mathbf{A}_{O'O}$ naturally, in such a way that O' is a source, and for every S a word on $\{a, b\}$, the edges Sa and Sb are positively oriented.

If, ϕ is already defined on S' and S, so that $S'S$ is an oriented edge, then,

$$\phi(SX) = \phi(S) + \alpha_S(X)(\phi(S) - \phi(S')),\ X \in \{a, b\}$$

where

$$\alpha_S : \{a, b\} \to \{-j, -j^2\}$$

is a *bijection.*

Therefore, ϕ is completely determined by, the "randonm variables" α_S, associated to any S (a word on a and b).

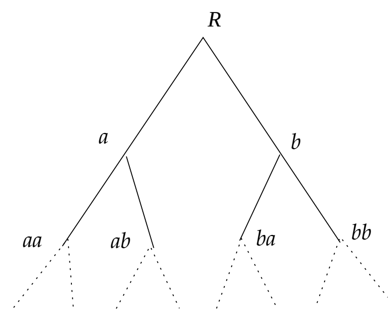

Figure 3 : Rooted tree

4.2.1. *Chain rule.* If $O' = S_{-1}, O = S_0, \dots S_n$ is a geodesic, i.e. a simple path in $\mathbf{A}_{O'O}$, and $w = \phi(O) - \phi(O')$, then,

$$\phi(S_n) = \phi(O') + w + \alpha_0 w + (\alpha_1\alpha_0)w + \dots + (\alpha_{n-1}\alpha_{n-2}.\dots.\alpha_0)w,$$

where $\alpha_i = \alpha_{S_i}$.

4.2.2. *Planar orientation, Canonical example.* For two given different complex numbers α and β, there is a canonical holomorphic function Φ with $\Phi(O') = \alpha$ and $\Phi(O) = \beta$. It is determined by taking a same α_S for all S, defined by:

$$\alpha_S(a) = -j,\ \alpha_S(b) = -j^2$$

This Φ can be characterized among holomorphic mappings by the fact that it preserves "2-dimensional" (or planar) orientation of the tree. To define it, imagine the tree naturally embeded in an oriented Euclidean plane. This induces an "orientation" on the tree, that is a choicee of positive angle between Sa, and Sb, for any $S \in \mathbf{A}$. Here, we decree that $\angle(Sa, Sb)$ is positive. In this case Φ actually preserves the orientation. It will be seen just below how to obtain all holomorphic mappings by means of Φ, and then that only Φ preserves orientation.

Summarizing these nice properties of Φ:

Fact 4.2. Φ *is holomorphic, (planar) orientation preserving, and* locally injective*, that is, at any vertex,* Φ *is injective on its immediate neighbourhood (i.e. on the* $\mathbf{Y}$ *around it), and is conformal (as defined in Remark 4.1).*

4.2.3. *Simply-transitive action of Aut.* Let $\mathcal{H}_{\alpha,\beta}$ be the space of holomorphic functions on $\mathbf{A}_{O'O}$ with prescribed (and distinct) values $\phi(O') = \alpha$ and $\phi(O) = \beta$.

Let $Aut(\mathbf{A})$ be the automorphism group of $\mathbf{A}$ or equivalently $\mathbf{A}_{O'O}$. It fixes O' and O, and for any S, there are elements of the group which exchange Sa and Sb. On the other hand, this exchange was the only one freedom in determining the holomorphic function, step by step, as explained above. This can be translated into the fact that, the right action by composition of $Aut(\mathbf{A}_{O'O})$ on $\mathcal{H}_{\alpha,\beta}$, is transitive.

On the other hand, this action is free. To see this, one verifies that Φ has a trivial stabilizer, that is no γ except the identity satisfies : $\Phi \circ \gamma = \Phi$. To prove it, one sees that, such a γ must fix both a and b. Therefore, at the next step, that is for words of length 2, the only non-trivial possible action of γ is to induce a transposition on $\{aa, ab\}$ or $\{ba, bb\}$. This means Φ takes a unique value on one on these sets, which contradicts the fact that Φ is locally injective (Fact 4.2). Now, the formal proof of the freedom of the action can be performed by induction.

Both $\mathcal{H}_{\alpha,\beta}$ and $Aut(\mathbf{A}_{O'O})$ are compact (when endowed with the compact-open topology, although the elements of $\mathcal{H}_{\alpha,\beta}$ are not bounded!). They are in fact topologically Cantor sets. One checks that: $\gamma \to \Phi \circ \gamma$ is a homoeomorphism.

4.2.4. *Action of the similarity group.* The similarity group SG of $\mathbb{C}$ acts (by the left) on the space of holomorphic functions for any space. They are mappings : $z \to \theta z + t$, where $\theta, t \in \mathbb{C}$, $\theta \neq 0$.

In our case, let $Hol = Hol(\mathbf{A}_{O'O})$ be the space of holomophic functions. This is a locally compact space (for the compact-open topology). Let Hol^* be the set of non-constant functions, that is, $Hol^* = Hol - \mathbb{C}$. Equivalently, Hol^* is the (disjoint) union of all the $\mathcal{H}_{\alpha,\beta}$, for $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$. Then, SG permutes the spaces $\mathcal{H}_{\alpha,\beta}$. In fact, a non-trivial element of SG can preserve no individual $\mathcal{H}_{\alpha,\beta}$. On the other hand, $Aut(\mathbf{A}_{O'O})$ acts by preserving each $\mathcal{H}_{\alpha,\beta}$. One then consider the produt action of $SG \times Aut(\mathbf{A}_{O'O})$. From previous developments, we get:

Fact 4.3. *The product action of $SG \times Aut(\mathbf{A}_{O'O})$ on Hol^* is simply transitive. Consequently, Hol^* is homeomorphic to $SG \times Aut(\mathbf{A}_{O'O})$.*

4.3. **Full tree T_3.** The full (bi-infinite) tree T_3 is obtained by gluing $\mathbf{A}_{O'O}$ and $\mathbf{A}_{OO'}$ along their respective edges $O'O$ and OO'. The group $Aut(\mathbf{A}) \times Aut(\mathbf{A})$ can be viewed as the subgroup of $Aut(T_3)$ fixing the edge $O'O$ (i.e. each of its extremities). With a little bit analysis, one gets:

Proposition 4.4. *An element of $Hol^*(T_3)$, that is a non-constant holomorphic function on T_3 is locally injective, more precisely it is conformal (as defined in Remark 4.1).*

Let $\mathcal{H}_{\alpha,\beta}(T_3)$, for $\alpha \neq \beta$, be the space of holomorphic functions taking values α and β on two adjacent vertices. The action of $Aut(\mathbf{A}) \times Aut(\mathbf{A})$ on it is simply transitive. The same is true for the action of the group $SG \times Aut(\mathbf{A}) \times Aut(\mathbf{A})$ on the space $Hol^*(T_3)$, which are therefore homeomorphic.

In contrast, the (transitive) action of $Aut(T_3)$ on $Hol^*(T_3)$ is not free. The stabilizer of a point is a (canonical) discrete group Γ acting freely on the tree T_3.

4.4. **Hexagonal tiling.** To given α and $\beta = \alpha + w$, one associates a hexagonal tiling of $\mathbb{R}^2$ (=$\mathbb{C}$), with α as a vertex, and w as an edge (trough α). The fundamental tile is the regular hexagon with vertices $\alpha, \alpha + w, \alpha + w - jw, \alpha + w - jw + j^2 w, \alpha + w - jw + j^2 w - w, \alpha + w - jw + j^2 w - w + jw$.

The tiling $\tau_{\alpha,\beta}$ is obtained by reflections along its edges of the fundamental hexagon. Up to similarity, we can restrict ourselves to $\tau = \tau_{0,1}$.

From §4.2.1, we get:

Proposition 4.5. *Let, $\phi \in Hol^*(T_3)$, that is a non-constant holomorphic function on T_3. Then, up to rescaling by a similarity of $\mathbb{C}$, we have:*

- *ϕ sends (the vertices of) T_3 onto (the vertices of) τ.*
- *In particular, for a geodesic (injective path) $\mathcal{C} = S_0, S_1, \ldots, S_n, \ldots$ in T_3, the image, $c = s_0, \ldots, s_n = \phi(S_n), \ldots$ is a path of τ.*

- The path c is locally injective, that is $s_i \neq s_{i+1} \neq s_{i+2}$.

- *Moreover, because of the local injectivity of ϕ, a ramification of paths in T_3 induces a ramification in τ. More exactly, if two geodesics $\mathcal{C} = S_1, \ldots, S_n, \ldots$ and $\mathcal{C}' = S'_1, \ldots, S'_n, \ldots$ bifurcate at i, that is, $S_j = S'_j$, for $j < i$, and $S_i \neq S'_i$, then, the same is true for their images, that is, $s_i \neq s'$.*
- *Summarizing: $\phi : T_3 \to \tau$ is the universal covering, and the discrete group Γ in Proposition 4.4 is the fundamental group of τ*

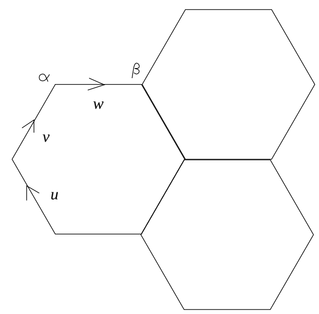

Figure 4 : Hexagonal tiling

4.4.1. *Locally injective random walk on τ.* Let $\mathcal{A}$ be the set of oriented edges of the fundamental hexagon of τ, which are labelled u, v, w, and then (necessarily) $-u, -v, -w$. Consider $\Omega = \mathcal{A}^{\mathbb{N}}$. Let $X_i : \omega = (\omega_i) \in \Omega \to \omega_i \in \mathbb{C}$. Consider (partial) sums $S_i(\omega) = \sum_{j=0}^{j=i} X_j(\omega)$. A sequence $S_0, S_1, \ldots S_n$ is supposed to represent a path of τ. However, not all ω's are allowed. For instance, if $S_0(\omega) = u$, then, in order that ω represent a walk in τ, $X_1(\omega)$ must equals $-u, v$ or $-w$. In general ω belongs to a subshift of finite type defined by means of an incidence matrix A^τ, with entries 0 and 1: a sequence ω is allowed, iff, $A^\tau_{\omega_i \omega_{i+1}} = 1$, for all $i \in \mathbb{N}$. Now, being furthermore a *locally injective* path in τ, leads to another subshift Σ^τ_{inj}. The new forbidden subsequences are of the form $y, -y$, where, $y \in \mathcal{A}$. If $\mathcal{A} = \{u, v, w, -u, -v, -w\}$ is identified, preserving order with $\{1, \ldots, 6\}$, then the incidence matrix of Σ^τ_{inj} is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

• Ergodic theory of the subshift Σ^τ_{inj}, growth of random variables S_n, are straightforwardly related to growth of holomorphic functions on T_3.

5. Proof of the graph case in Theorems 1.2 and 1.6

We have the following straightforward generalization of Lemma 3.2

Fact 5.1. *Let δ_1, δ_2 be given. Consider the set of equations on unknowns $\delta_3, \ldots, \delta_n$:*

$$\sum_{i=3}^{n} \delta_i^p + (\delta_1^p + \delta_2^p) = 0,$$

for $p \in \{1, \ldots, N\}$, where N is an integer.

If $N \leq n-2$, then a solution exists. It is unique up to permutation (of the $\delta_i, i \geq 3$) if $N = n-2$. In opposite, if $N > n-2$, then, in order that solutions exist, δ_1 and δ_2 must vanish, and the solution in this case is trivial : $\delta_i = 0$, $\forall i$.

This yields as a corollary the claim that a holomorphic function which satisfies $(*)_N$ where N is the (maximal) valency of the graph X is constant. In particular, in the case of T_3 only constant functions satisfy $(*)_3$, so all other holomorphic functions satisfy $(*)_2$ but not $(*)_3$.

A harmonic morphism $X \to \mathbb{C}$, must satisfy $(*)_\infty$. Only constant functions are so (say, assuming the graph has a finite valency).

This proves all the discrete content of Theorems 1.2 and 1.6.

5.0.2. *N-holomorphic functions.* The previous fact suggests to introduce N-holomorphic functions as those satisfying $(*)_N$. For instance a theory of $(*)_N$ holomorphic functions on T_{N+1}, the tree of valency $N+1$ may be developed in a same way as holomorphic functions on T_3. The situation is a little bit complicated. For instance, instead of the hexagonal tiling, we get a picture with other kinds of piecewise lines. (See the following figures in the cases $N = 2, 3, 4, 5$)

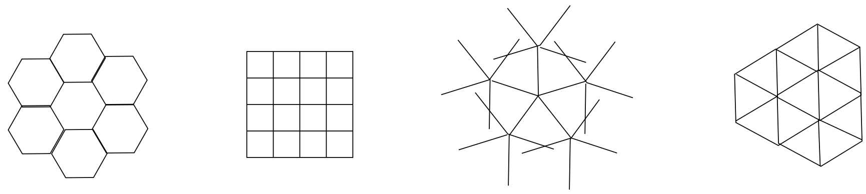

Figure 5 : N-holomorphic functions

6. The graph Tr_3

The graph Tr_3, is a kind of 3-valenced tree, where vertices are replaced by triangles, which explains our notation here.

More precisely Tr_3 is a homogeneous graph with valency 4 and (minimal) cycles of length 3.

It is obtained from a countable family of triangles, where, to each triangle is glued exactly another one at each vertex.

It can also be considered as the "dual graph" of the tree T_3, that is, the graph which vertices are the edges of T_3, and two of them are joined by an edge (in Tr_3) if the have a common vertex (in T_3). A concrete model is obtained by taking middles of the edges of T_3 and drawing segments joining middles of adjacent edges.

In the sequel, we will see, analogously to the case of T_3, that a holomorphic function, is constructed following some dynamics, when it is given at vertices of some triangle. As in the case of T_3, a nuisance but also a richness of the dynamics comes from its non-deterministic character, that is, the switch choice. However, with respect to this point, the case of Tr_3 is highly more complicated. Our attempt here is just to roughly study this dynamics, which we think is worthwhile to investigate, at least because it is natural!

6.1. **Dynamics on triangles.**

6.1.1. *Starting-up.* Let OAB be a triangle in Tr_3, and OCD and adjacent one, with a common vertex O. If ϕ is holomorphic, and is given at O, A and B, then using Fact 3.3 in a same way as in the case of T_3, one deduces the values of ϕ at C and D, up to a switch.

Let's introduce notations which will be kept along this §. We denote:

$$p = \phi(O), e = \phi(A) - \phi(O), f = \phi(B) - \phi(O), u = \phi(C) - \phi(O), v = \phi(D) - \phi(O)$$

Here p, e and f are given. We infer from Fact 3.3, that u and v can be computed, but up to a switch. Indeed, this reduces to solving the following equation on u and v, where e and f are supposed known:

$$u + v + (e + f) = u^2 + v^2 + (e^2 + f^2) = 0$$

In other words, after elementary algebraic manipulation:

Fact 6.1. *u (or equivalently v) is a solution of the second order equation:*

$$u^2 + (e + f)u + (e^2 + f^2 + ef) = 0$$

(In the sequel, we will choose, arbitrary solutions, $\psi_1(e, f)$ and $\psi_2(e, f)$, see below).

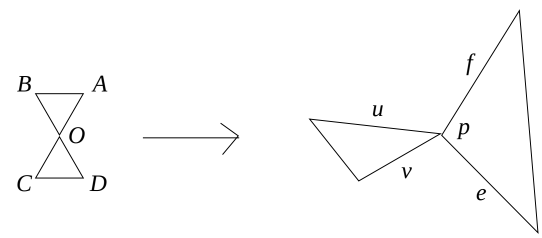

Figure 6 : Two triangles

6.1.2. *Rough formulation.* Now, once ϕ is determined on OCD one uses data on this triangle to extend ϕ on an adjacent triangle, say that with vertex C. The data are thus $(\phi(C), \phi(O) - \phi(C), \phi(D) - \phi(C))$, or equivalently $(p - u, -u, -u + v)$. Therefore, we are led to study the transformation:

$$M : (p, e, f) \to (p - u, -u, -u + v)$$

The difficulty comes of course from that this is not an univalued transformation. Indeed M is constructed by means of the 2-valued mapping:

$$I : (e, f) \in \mathbb{C}^2 \to (u, v) \in \mathbb{C}^2$$

In order to get univalued mappings, we choose, in an arbitrary way, solutions $\psi_1, \psi_2 : \mathbb{C}^2 \to \mathbb{C}$ as in Fact 6.1. Actually, at this stage, we do not mind on any regularity of these mappings., but the only "algebraic" condition is that, this gives rise to an involution, that is, $\psi = (\psi_1, \psi_2)$ is an involution.

Therefore, we get two "sections" for M:

$$M_1 : (p, e, f) \to (p - \psi_1(e, f), -\psi_1(e, f), -\psi_1(e, f) + \psi_2(e, f)),$$

$$M_2 : (p, e, f) \to (p - \psi_2(e, f), -\psi_2(e, f), -\psi_2(e, f) + \psi_1(e, f))$$

Summarising:

Fact 6.2. *If the image of the triangle (O, OA, OB) by ϕ is (p, e, f), then the image of (C, CO, CD) and (D, DO, DC) are, in order, either, $M_1(p, e, f)$, $M_2(p, e, f)$, or inversely $M_2(p, e, f)$, $M_1(p, e, f)$.*

6.1.3. *The involution.* An important part of dynamics is contained in the 2-valued mapping, $I : (e, f) \in \mathbb{C}^2 \to (u, v) \in \mathbb{C}^2$. It is an involution (as a 2-valued mapping).

One way to let this being a "standard" mapping, is to consider the quotient space $\mathbb{C}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts by $(e, f) \to (f, e)$. This gives rise to a well defined (continuous) involution $\overline{I}$ of $\mathbb{C}^2/\mathbb{Z}_2$.

In order to see a simple picture of this, it is useful to projectivize all things. In this case, the $\mathbb{Z}_2$ action on $\mathbb{C}P^1$ becomes a π rotation, around, north and south poles, say. The quotient space is still, topologically, a 2-sphere.

It is suggestive to think about $\overline{I}$ as a holomorphic mapping of the "orbifold" $\mathbb{C}^2/\mathbb{Z}_2$ (or equivalently $\mathbb{C}P^1/\mathbb{Z}_2$). This seems to not be so right, since for instance, $\overline{I}$ does not preserve the singular locus of orbifolds. Anyway, the true dynamics which interests us, that is, M_1 and M_2, are not compatible with tacking this quotient.

Maybe, the induced mapping $\mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}_2$, could be thought, more naturally, as holomorphic. However, having a different source and target spaces, deprives one to perform dynamics (i.e. to iterate).

6.1.4. *Catastrophe.* The "singular locus" of I is the set of (e, f), where equation 3.3 has a double root, that is $u = v$. It consists of the two complex lines $\mathcal{S}$ and $\mathcal{N}$ in $\mathbb{C}^2$ generated by $(1 - i\sqrt{2}, 1 + i\sqrt{2})$ and $(1 + i\sqrt{2}, 1 - i\sqrt{2})$. In the projective plane $\mathbb{C}P^1$, this corresponds to two points, $\{S, N\}$, say. One then hopes that ψ_1 and ψ_2 (introduced in §6.1.2) can be continuously defined outside $\mathcal{S} \cup \mathcal{N}$, or equivalently on $\mathbb{C}P^1 - \{S, N\}$. It turns out that this is not possible : despite $\psi_1 \neq \psi_2$ along $\mathbb{C}P^1 - \{S, N\}$, the *real* line bundle that they define (the fiber above (e, f) is the real line joining $\psi_1(e, f)$ and $\psi_2(e, f)$) is not (topologically) trivial, i.e. non-orientable (on $\mathbb{C}P^1 - \{S, N\}$).

6.2. **Random dynamics.** The graph Tr_3 can be encoded as follows. We have a central triangle Δ_0. We denote by $1, 2, 3$ its vertices. Any triangle Δ in Tr_3 is obtained by specifying a vertex i which indicated the "direction" of Δ, together with a word of the free monoid F_2 on two letters $\{a, b\}$ (which may be empty $\emptyset$). Therefore, any $\Delta \neq \Delta_0$ is encoded by (l, i), where $l \in F_2$, and $i \in \{1, 2, 3\}$.

For instance, in the direction 1, we have:

$$\Delta_0, (\emptyset, 1), ((a, 1), (b, 1)), ((aa, 1), (ba, 1), (ab, 1), (bb, 1)), \ldots$$

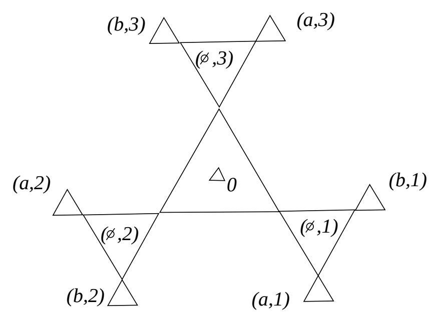

Figure 7

Actually, one must deal with marked triangles. For instance, the central (geometric) triangle Δ_0 gives rise to three marked triangles (Δ_0, i), where $i \in \{1, 2, 3\}$ indicates the vertex.

Each triangle (l, i) gives rise to 3 marked triangles, one for each vertex. Firstly, a marked triangle $(l, i, 0)$, where the marked vertex is that sheared by its (previous) generating triangle. Now, imagine Tr_3 naturally embed in an oriented Euclidean plane. This induces an orientation (i.e. an order on the edges) of the marked triangle $(l, i, 0)$. We get therefore two marked triangles $(l, i, -)$ and $(l, i, +)$.

A generic marked triangle will be denoted $\overline{\Delta}$ and its underlying (non-marked) triangle is denoted Δ.

One can speak of $c\Delta$, for c a letter ($\in \{a, b\}$). Also, one defines, in a natural way, $c\overline{\Delta} = c(l, i, \epsilon) = (cl, i, \epsilon)$, for $\epsilon \in \{0, -, +\}$ a sign. In other words, the action of F_2 is trivial on the marking.

Exactly as in §4.2, we have:

Proposition 6.3. *Let ϕ be a holomorphic function on Tr_3. For any triangle Δ, there is α_Δ, a bijection $\{a, b\} \to \{M_1, M_2\}$, such that, $\phi(c\Delta) = \alpha_\Delta(c)(\Delta)$, where c is a letter $\in \{a, b\}$. The same action is induced on marked triangles, that is, $\phi(c\overline{\Delta}) = \alpha_\Delta(c)(\overline{\Delta})$.*

6.2.1. *Special holomorphic functions.* As in the case of T_3, some holomorphic functions appear special. They are obtained, in a deteministic way, that is all the α_Δ are the same, for example, $\alpha_\Delta(a) = M_1, \alpha_\Delta(b) = M_2$ (for any Δ). Note however that this depends on the choice of M_1 and M_2, or equivalently that of ψ_1 and ψ_2 (§6.1.2)

6.2.2. *Action of $Aut(Tr_3)$.* Let Aut_0 denote the stabilizer of Δ_0 in $Aut(Tr_3)$, the automorphism group of Tr_3. Applying elements of Aut allows one to restore all randomness in the construction of holomorphic functions, that is the choice of the α_Δ. From it one infers that, two holomorphic functions taking the same values on Δ_0 are equivalent up to composition in Aut_0.

Now, if one considers the full action of $Aut(Tr_3)$, one gets:

Fact 6.4. *Two holomorphic functions ϕ and ϕ' are equivalent in $Aut(Tr_3)$, iff, they take the same values on two triangles Δ and Δ'*

6.2.3. *Space of triangles. Orbital structure.* The space of marked (oriented) triangles is $\mathcal{T} = \{(p, e, f) \in \mathbb{C}^3, (e, f) \neq (0, 0)\}$ (as was said previously, degenerate triangles, i.e. with $e = 0$ or $f = 0$, are also considered). We have seen, how holomorphic functions induce, a kind of random dynamics on $\mathcal{T}$. Consider the relation $\sim$ defined by

$$\boldsymbol{\Delta} \sim \boldsymbol{\Delta'} \iff \exists \phi \text{ holomorphic}, \Delta, \Delta' \text{ triangles of } Tr_3, \boldsymbol{\Delta} = \phi(\Delta), \boldsymbol{\Delta'} = \phi(\Delta')$$

In other words, thanks to the Fact above: $\boldsymbol{\Delta} \sim \boldsymbol{\Delta'}$, iff, there is ϕ holomorphic, $\gamma \in Aut(Tr_3)$, such that $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta'}$ are images of (the central triangle) Δ_0 by ϕ and $\phi \circ \gamma$, respectively.

Previous developments show that $\sim$ is a well defined *equivalence* relation on $\mathcal{T}$ (despite randomness, and arbitrary choose of ψ_1, ψ_2...).

It is worthwhile to investigate ergodic theory of this relation!

6.2.4. *Action of the similarity group, Conformal triangles.* All involved dynamical notions I, M_1, M_2, $\sim$... commute with the action of the similarity group SG of $\mathbb{C}$. The marked vertex of the triangle can be identified under this action with $0 \in \mathbb{C}$. It remains to consider pairs $(e, f) \in \mathbb{C}^2 - (0, 0)$, up to homothety. This is exactly $\mathbb{C}P^1$, which is interpreted as the space of conformal triangles (allowed to be degenerate).

In particular, $\sim$ passes to an equivalence relation $\approx$ on $\mathbb{C}P^1$.

Let us observe here that an adapted parallel construction in the case of T_3, yields an equivalence relation $\approx$ which is trivial. Here, our $\approx$ is far from being so: not only the dynamics on triangles is not trivial, but even their conformal type is strongly transformed.

6.3. **Language of correspondences.** Denote $(x, y, z) = (p-u, -u, -u+v)$. Then, $(p, u, v) = (x + y, -y, -y + z)$. Therefore, the "graph" of the multi- valued transformation M, is the subset of $(p, e, f; x, y, z) \in \mathbb{C}^3 \times \mathbb{C}^3$, such that:

$$\begin{aligned} e + f - y + (-y + z) &= 0 \\ e^2 + f^2 + y^2 + (-y + z)^2 &= 0 \\ x - (p + y) &= 0 \end{aligned}$$

The space of solutions $\mathcal{S}$, is a 3-dimensional quadric contained in a 4-dimensional linear subspace. We have surjective induced projection $\pi_1 : \mathcal{S} \to \mathbb{C}^3$ and $\pi_2 : \mathcal{S} \to \mathbb{C}^3$. (Actually, instead of $\mathbb{C}^3$, one may better consider the space of triangles $\mathcal{T}$, and $\mathcal{S} \subset \mathcal{T} \times \mathcal{T}$). The projection π_1 and π_2 are branched converings. With all these objects, we are bringing out a natural situation leading to the useful notion of correspondences (see [2] for fundamental references). The point is to consider the multi-valued dynamics: $\boldsymbol{\Delta} \to \pi_2(\pi_1^{-1}(\boldsymbol{\Delta}))$.

6.3.1. *Projective version.* As above, one can see what happens for conformal triangles by taking quotient under the action of the similarity group. More precisely, we consider the quotient of $(\mathbb{C} \times (\mathbb{C}^2 - 0)) \times (\mathbb{C} \times (\mathbb{C}^2 - 0))$ by the product action of $SG \times SG$ (recall that SG is the group of similarities of $\mathbb{C}$). We get $\mathbb{C}P^1 \times \mathbb{C}P^1$, and

inside it, corresponding to $\mathcal{S}$, a quadric $\mathcal{C}$, which thus determines a correspondence of $\mathbb{C}P^1$.

6.3.2. *Growth, Simulation.* As was said, we restrict ourselves in this case to formulation rather than a systematic study of the dynamics. In order to have an idea, one can implement the multi-valuated mapping : $M(p,e,f) = (\frac{2p+(e+f)-\sqrt{-3(e^2+f^2)-2ef}}{2}, \frac{(e+f)-\sqrt{-3(e^2+f^2)-2ef}}{2}, -\sqrt{-3(e^2+f^2)-2ef})$ and gets, as example the following two pictures :

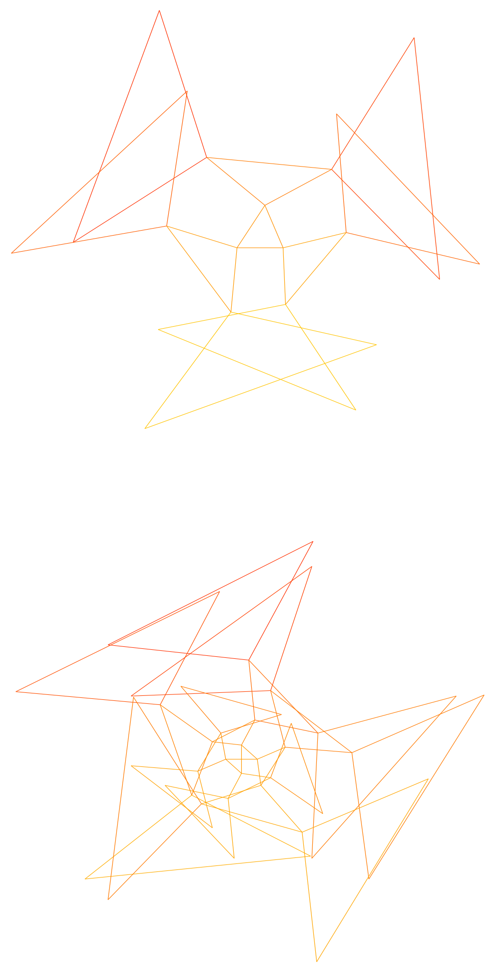

Figure 8 : Images of balls in Tr_3 of radius 3 and 5 by holomophic functions

7. Remarks on other graphs

7.0.3. *Graphs of valency 3.* Any graph X of valency 3 is (universally) covered by T_3. Therefore any holomorphic function on X gives rise to a holomorphic function on T_3, which is therefore fully understood thanks to Theorem 1.7. We straightforwardly deduce from it the following:

Theorem 7.1. *Let X be a graph of valency 3. If X is the hexagonal tiling, then any holomorphic function $X \to \mathbb{C}$ is the restriction of a similarity.*

If X has a cycle of length ≤ 5, then, any holomorphic function on X is constant (here we tacitly assume X is connected). In fact, in general, holomorphic functions are trivial unless X covers the hexagonal tiling.

As example, the cayley graph of $PSL(2, \mathbb{Z})$ has no non-constant holomorphic functions.

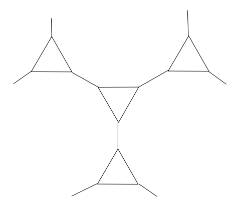

Figure 9 : Cayley graph of $PSL(2, \mathbb{Z})$

7.0.4. *Cayley graph of $\mathbb{Z}^n$.* Consider the cayley graph X_n of $\mathbb{Z}^n$ (associated to the canonical basis together with its opposite), that is, the standard lattice in $\mathbb{R}^n$.

For $n = 1$, harmonic functions are affine : $n \mapsto a + bn$ and holomorphic functions are constant.

Identify X_2 with $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ embeded in $\mathbb{C}$. One easily verifies that, only if $k \leq 3$, the function $z \mapsto z^k$ is harmonic on X_2. On the other hand , the mappings $z \mapsto az + b$, and similarly $z \mapsto a\bar{z} + b$, are holomorphic. It is very likely that these are all holomorphic functions on X_2?

7.1. **Non-rigidity, Trees of higher valency.** For a tree T of valency ≥ 4, and two adjacent vertices s_0 and s_1, on which a function ϕ takes values z_0 and z_1, there is at least 1 degree of freedom in extending holomorphically, progressively ϕ to neighbouring vertices. Assume, to simplify notation, the valency equals 4 and let s_2, s_3 and s_4 the other neighbours of s_0. The associated values $z_i = \phi(s_i)$, for $i = 2, 3, 4$, satisfy:

$$\sum_{i=2}^{i=4} \delta_i = -\delta_1$$
$$\sum_{i=2}^{i=4} \delta_i^2 = -\delta_1^2$$

where $\delta_i = z_i - z_0$. Only δ_1 is given, thus, we have 2 equations and 3 unknowns, and hence, a 1-dimensional complex space of solutions.

7.1.1. *Bounded holomorphic functions.* We will now observe that with this freedom, we can choose ϕ to be bounded (and non-constant). (Even not formally stated previously, this is not possible for T_3 and Tr_3). To do this, we show that, there is a positive real number $r < 1$, such that, for any δ_1, the above equations admit solutions satisfying $|\delta_i| \leq r|\delta_1|$, for $i = 2, 3, 4$. (With this, one constructs a holomorphic function ϕ having an exponentially decreasing oscillation along edges, and hence bounded). By homogeneity, it suffices to consider the case $\delta_1 = -1$. To get a solution in this case, we write $\delta_2 = r_1 e^{i\theta}$, $\delta_3 = \bar{\delta_2} = r_1 e^{-i\theta}$ and $\delta_4 = r_2$ real and positive. We claim that equations:

$$2r_1 \cos\theta + r_2 = 1, \; 2r_1^2 \cos 2\theta + r_2^2 = -1$$

admit solutions with $0 < r_1, r_2 < 1$. This is done by elementary analysis around $r_1 = r_2 = 1$, and $\theta = \frac{\pi}{2}$.

Remark 7.2. As was suggested in §5, a theory parallel to that of T_3, may be developed for a tree T_N of valency N, if one considers "$(N-1)$-holomorphic functions". This would surely give beautiful pictures.

8. The conjugate part Problem

Let us recall the problem. We consider a *real* function f on a graph X, and we ask, when (and how) it has a conjugate part, that is another real function g, such that $\phi = f + ig$ is holomorphic? Actually, we have finally give up to systematically investigate this problem in the present article, since of unexpected subtleties. Only partial results will be exposed here. To mention a rough conclusion, we say that also here a random dynamics is in order, and as it is natural to expect, the problem has some cohomological flavour.

8.0.2. *Conditions.* It turns out that the same conditions, stated in the Riemannian case, as explained after Definition 1.1, hold in the discrete case. For any vertex s, g satisfies:

$$\| \nabla_s f \| = \| \nabla_s g \|, \ \langle \nabla_s f, \nabla_s g \rangle = 0, \ \langle \nabla_s g, (1, \dots, 1) \rangle = 0$$

The last condition expresses the fact that g is harmonic. If s has valency n, these conditions mean exactly: $\nabla_s g$ must belong to $\mathcal{C}_s^f$, a $(n-3)$-sphere of radius $\| \nabla_s f \|$ in $\mathbb{R}^n$.

From this "freedom", say when $\mathcal{C}^f$ has higher dimension, one may ask if f is allowed to have more than one conjugate part (up to constant). It turns out this is possible, for instance in the case of the (homogeneous) tree of valency 4. Notice however, that because of the norm gradient equality, the space of conjugates (of a given f) is compact (up to constants).

8.0.3. *Non-existence examples.* On the other hand, on the same tree (of valency 4), there are harmonic functions with no conjugate part. For an example (see Figure 10), consider A a vertex, B, C, D and E its neighbours. Consider the sub-tree $\mathcal{A}$ consisting of A, B, C, D, E and the 3 remaining vertices of each B and C.

Take f to be everywhere 0 on $\mathcal{A}$, except on D and E where it takes two opposite non vanishing values, say 1 and -1.

Suppose a conjugate g exists. Adding a constant, we can assume $g(A) = 0$. The norm gradient equality implies that, like f, g vanishes on all neighbours of B and C.

On the other hand, writing the conjugate part equations on A, we get solutions which must vanish on D and E, but not on B and C, which is a contradiction. Observe here that f can be easily extended (not uniquely) to a harmonic function on all the tree.

One may believe the "degeneracy" of f, that is the vanishing of many of its oscillations and gradients, was the responsible on non-existence of its conjugate part. This is not true, all harmonic functions sufficiently near f have no conjugate part as well. Indeed, having a conjugate part is a closed condition. This follows from the compactness mentioned above (due the the norm gradient equality).

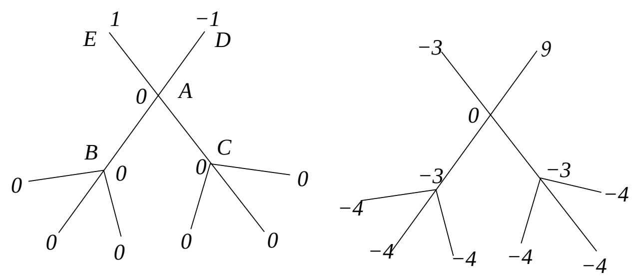

Figure 10 : No conjugate part

8.0.4. *An existence result.* From the classification of holomorphic functions on T_3, one infers that a harmonic function has conjugate part if and only if its gradient has a constant norm (throughout T_3) (Actually, this corollary is equivalent to the classification itself). Our principal existence result is that this condition, norm constancy of the gradient, is sufficient for trees of higher valences. This is, by no means, a necessary condition, but seems nevertheless, natural, and gives an interesting class of examples.

Theorem 8.1. *Let f be a harmonic on a (homogeneous) tree X, of (finite) valency ≥ 4, and assume its gradient has a constant norm, say, $\| \nabla_s f \| = 1$, $\forall s \in X$. Then f has a conjugate part. There is $(\nu - 3)$ degrees of freedom (at each vertex) in choosing this conjugate part, where ν is the valency. The space of conjugates of f is compact.*

Proof. Choose $O \in X$, and for $r \in \mathbb{N}$, let B_r be the (closed) ball centred at O of radius r (i.e. the set of vertices that can be reached from O by a sequence of at most r edges). Consider also the sphere S_r, the boundary of B_r.

Trees are characterized by the following "strict convexity" property of B_r: if $s \in S_r$, then, only one neighbour $p(s)$ of s lies in B_r, in fact in $B_r - S_r$. All the others are outside B_r, in fact they belong to S_{r+1}. Furthermore a vertex of S_{r+1} is adjacent to exactly one vertex of S_r.

Assume g is defined on B_r, and let us show how to extend (with some freedom) it to B_{r+1}. Let $s \in S_r$. Then, g is already defined on exactly one neighbour $p(s) \in S_{r-1}$ of s.

In order, to extend g to other neighbours ($\in S_{r+1}$), we just have to give $\nabla_s g$. Actually, exactly one component, say a_1, of $\nabla_s g$ is already given, that corresponding to the edge $p(s) - s$. The gradient $\nabla_s g$ must belong to the sphere $\mathcal{C}_s^f$. The question reduces then to the fact that this sphere $\mathcal{C}_s^f$ contains a vector with first projection a_1 (again to be precise, by first projection, we mean that corresponding to the edge $p(s) - s$). The proof of the theorem will be completed thanks to the following lemma. Its significance is that, the spheres $\mathcal{C}_s^f$ and $\mathcal{C}_{p(s)}^f$ have exactly the same projection on the coordinate corresponding to the edge $p(s) - s$. $\square$

Lemma 8.2. *Let $n \geq 4$, $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n \setminus \{0\}, \sum \delta_i = 0$, $\sum \delta_i^2 = 1$. Consider the sphere*

$$\mathcal{C} = \{a = (a_1, ..., a_n) \in \mathbb{R}^n, \sum a_i \delta_i = 0, \sum a_i^2 = 1, \sum a_i = 0\}$$

Then the image of $\mathcal{C}$ by the projection on the first factor of $\mathbb{R}^n$ is the segment $[-\alpha, \alpha]$ where $\alpha = \sqrt{\frac{n-1}{n} - \delta_1^2}$. In particular, this depends only on δ_1.

Before proving it let us make some remarks about this lemma. Firstly, it was stated in the case $\| \delta \| = 1$, but, obviously, a general formula is available for any constant norm.

We also notice that a similar result is true for $n = 3$, the projection in this case consists of two points : $\pm\sqrt{\frac{2}{3} - \delta_1^2}$. The theorem itself is true for T_3, in fact as was said above, it is optimal in this case. Observe however, that the theorem does not extend to the (trivial) case of the 2-valenced tree $T_2 (\approx \mathbb{Z})$.

Finally, observe that, our second example of a harmonic function without conjugate part (see Fig. 10) corresponds to the case where $\alpha = 0$ (i.e. $\delta_1 = \sqrt{3}/2$ and $\delta_2 = \delta_3 = \delta_4 = -\sqrt{3}/6$).

Proof. Consider the mapping $\pi : a \mapsto a_1$. The projection of the gradient of π on the hyperplane $H = \{(x_1, ..., x_n) \in \mathbb{R}^n, \ \sum x_i = 0\}$ is collinear to $v = (1-n, 1, ..., 1)$, and so if a is a critical point of π restricted to the submanifold $\mathcal{C}$, then there exist α, β so that $a = \alpha v + \beta\delta$. ¿From the fact that a is orthogonal to δ, we infer that : $\beta = -\alpha < v, \delta >$. But, since $\sum \delta_i = 0$, we have $< v, \delta >= -n\delta_1$. Finally, we have $a = \alpha((1-n) + n\delta_1^2, \ 1 + n\delta_1\delta_2, ..., 1 + n\delta_1\delta_n)$. Thus, $\| v - < v, \delta > \delta \|^2 = n(n - 1 - n\delta_1^2)$, and hence, $a_1^2 = \frac{n-1}{n} - \delta_1^2$. □

Remark 8.3. (Canonical conjugate?) The conjugate part problem is sub-determined: solutions are not unique (when they exist). It is interesting to define ways of selecting unique solutions. Maybe, by means of a companion variational problem?

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